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On topological chaos in the Robinson–Solow–Srinivasan model

M. Ali Khan[a,*], Tapan Mitra[b]

[a] *Department of Economics, The Johns Hopkins University, Baltimore, MD 21218, United States*
[b] *Department of Economics, Cornell University, Ithaca, NY 14853, United States*

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Abstract

Under a sufficient condition to ensure a unique optimal program, the theory of turbulence in non-linear dynamical systems allows us to exhibit an instance of (topologically) chaotic optimal behavior in a two-sector model with irreversible investment due to Robinson–Solow–Srinivasan.

Keywords: RSS model; Two-sector model; Optimal program; Modified golden-rule stock; Continuous policy functions; Turbulence; Topological chaos

JEL classification: D90; C62; O21

1. Introduction

In a stimulating paper whose full implications have not yet been exhausted, Stiglitz (1968) studied the Ramsey optimal growth problem in a model due to Solow, Srinivasan and Robinson; henceforth the RSS model.[1] In a setting with continuous time and a linear felicity function, and without *any* restrictions on the discount factor[2], Stiglitz appealed to Pontryagin's maximum principle to characterize the optimal

* Corresponding author. Tel.: +410 516 8545; fax: +410 516 7600.
E-mail addresses: akhan@jhu.edu (M.A. Khan), tm19@cornell.edu (T. Mitra).

[1] For references and geneology, see Khan and Mitra (2005).

[2] This includes the unit value for the discount factor, thereby allowing Stiglitz to consider both the discounted and undiscounted cases in one sweep, with the modified golden-rule stock reduced to the golden-rule stock in the undiscounted setting; see Section 2 below, and Khan and Mitra (2005, 2004b) for additional discussion.

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program as one that monotonically converges to a modified golden-rule stock. In recent work, the authors reformulate the RSS model in discrete time, and show that his results do not universally hold in the sense that, under specific parametric regimes, the optimal program does not satisfy the monotonicity property, and could cycle even in the undiscounted case; see Khan and Mitra (2005, 2004a,b). In the light of this work, a natural question arises as to whether optimal behavior in the RSS model can exhibit topological chaos. In this letter, we answer this question in the affirmative.

We work with a special case of the RSS model in which there is only one type of machine, as opposed to the many-machines, multi-sectoral setting considered in Stiglitz (1968) and Khan and Mitra (2005, 2004b). If the primary objective is to generate complicated dynamics from simple deterministic models, the results reported here bring out in a dramatic way that this special case of the two-sector model, one in which machines are not needed to produce machines, suffices for the purpose at hand. It is however of interest that the model, characterized as it is by only three positive real numbers (a, d, ρ), respectively specifying the labor–capital ratio, depreciation rate and the discount factor[3], does not fall within the canonical setting considered in the work of Mitra (1996) and Nishimura and Yano (1996) that furnishes an "exact" bound for the existence of period-three cycles as a universal constant [4]. As such, their results cannot be directly applied, but as in these papers, we work with the "value-loss" approach of McKenzie (2002)[5], and with "small" discount factors.

A final methodological observation. We present our result under a certain parametric regime, appealing to a result in the theory of turbulence. Our application highlights the fact that we do not appeal to *any* unimodal property of the optimal policy function, but only to its continuity [6]. Under the restriction that $\rho < a$ (a requirement that the discount factor be "small" given the parameterization[7] $\xi \equiv (1/a) - (1-d) > 1$ (so that $a < 1/(2-d)$), this result is a consequence of Berge's maximum theorem.

We hope that the results and the techniques we elaborate will generate interest in the RSS model and also be of use in further work on other models.

2. The two-sector version of the RSS model

A single consumption good is produced by infinitely divisible labor and machines with the further Leontief specification that a unit of labor and a unit of a machine produce a unit of the consumption good. In the investment-goods sector, only labor is required to produce machines, with $a > 0$ units of labor producing a single machine. Machines depreciate at the rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available in each time period $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. Thus, in the canonical formulation surveyed in McKenzie (1986, 2002), the collection of production

[3] As specified in the section below, this is after normalization of the other parameters of the model. Also, d and ρ are obviously bounded above by one.

[4] The derivation of this remarkable "universal" constant, $\left[\left(\sqrt{5}-1\right)/2\right]^2$, uses, in particular, assumptions A1, A4 and A7 in Mitra (1996) and assumption A2 in Nishimura and Yano (1996). These do not hold for our model and suggest an open question which we do not pursue in this brief note.

[5] This is most succinctly expressed by the statement that the planner loses by deviating from certain price-supported activities; see Section 3 below. As is well-known, the approach derives from Radner's 1961 paper; see McKenzie (2002; p. 245).

[6] The pioneering investigations of chaos in economics considered unimodal maps, as in Grandmont's (1986) exposition; also see Block and Coppel (1992; Preface).

[7] See Khan and Mitra (2004a) for a comprehensive discussion of the importance of ξ for the RSS model.

plans (x, x'), the amount x' of machines in the next period (tomorrow) from the amount x available in the current period (today), is given by the *transition possibility* set: $\Omega = \{(x,x')\in\mathbb{R}^2_+ : x' - (1-d)x \geq 0$, and $a(x'-(1-d)x)\leq 1\}$, where $z\equiv(x'-(1-d)x)$ is the number of machines that are produced, and $z\geq 0$ and $az\leq 1$ respectively formalize constraints on reversibility of investment and the use of labor. For any $(x, x')\in\Omega$, one can consider the amount y of the machines available for the production of the consumption good, leading to a correspondence $\Lambda:\Omega\rightarrow\mathbb{R}_+$ with $\Lambda(x,x') = \{y\in\mathbb{R}_+ : 0\leq y\leq x$ and $y\leq 1-a(x'-(1-d)x)\}$. Welfare is derived only from the consumption good and is represented by a linear function, normalized so that y units of the consumption good yields a welfare level y. A *reduced form utility function*, $u:\Omega\rightarrow\mathbb{R}_+$ with $u(x, x')=\max\{y\in\Lambda(x, x')\}$ indicates the maximum welfare level that can be obtained today, if one starts with x of machines today, and ends up with x' of machines tomorrow, where $(x, x')\in\Omega$. Intertemporal preferences are represented by the present value of the stream of welfare levels, using a discount factor $\rho\in(0, 1)$.

An *economy E* consists of a triple (a, d, ρ), and the following concepts apply to it. A *program* from x_o is a sequence $\{x(t), y(t)\}$ such that $x(0)=x_o$, and for all $t\in\mathbb{N}$, $(x(t), x(t+1))\in\Omega$ and $y(t)=\max\Lambda((x(t), x(t+1))$. A *program* $\{x(t), y(t)\}$ is simply a program from $x(0)$, and associated with it is a gross *investment sequence* $\{z(t+1)\}$, defined by $z(t+1)=(x(t+1)-(1-d)x(t))$ for all $t\in\mathbb{N}$. It is easy to check that every program $\{x(t), y(t)\}$ is bounded by $\max\{x(0),1/ad\}\equiv M(x(0))$, and so: $\sum_{t=0}^{\infty}\rho^t u(x(t),x(t+1))<\infty$. A program $\{\bar{x}(t), \bar{y}(t)\}$ from x_o is called *optimal* if: $\sum_{t=0}^{\infty}\rho^t u(x(t),x(t+1))\leq\sum_{t=0}^{\infty}\rho^t u(\bar{x}(t),\bar{x}(t+1))$ for every program $\{x(t), y(t)\}$ from x_o. A program $\{x(t), y(t)\}$ is called stationary if for all $t\in\mathbb{N}$, we have $(x(t), y(t))=(x(t+1), y(t+1))$. A *stationary optimal program* is a program that is stationary and optimal.

3. On the modified golden-rule

A distinctive feature of the RSS model with discounting is that the modified golden-rule stock is unique and is independent of the discount rate. This result is comprehensively discussed in Khan and Mitra (2004b); we state it without proof for the simpler (one machine-type) setting to lay the groundwork for our analysis.

Lemma 1. *There is $(\hat{x},\hat{p})\in\mathbb{R}^2_+$ such that $(\hat{x}, \hat{x})\in\Omega$ and: $u(\hat{x}, \hat{x})+(\rho-1)\hat{p}\hat{x}\geq u(x, x')+\hat{p}(\rho x'-x)$ for all $(x, x')\in\Omega$. Further, $\hat{x}$ and $\hat{p}$ are uniquely given by: $\hat{x}=1/(1+ad)$, $\hat{p}=1/(1+\rho\xi)$.*

Lemma 1 leads us to define the value-loss of a production plan $(x, x')\in\Omega$ relative to $(\hat{x}, \hat{p})$, and rewrite it in an amenable form, through the use of the identity $a(1+\xi)=(1/\hat{x})$, as:

$$\delta^{\rho}(x,x') = (\rho\hat{p}/a) - u(x,x') - \hat{p}(\rho x' - x) \tag{1}$$

If $\{x(t), y(t)\}$ is a program, then $(x(t), x(t+1))\in\Omega$ for $t\in\mathbb{N}$, so that, denoting $\delta^{\rho}(x(t), x(t+1))$ by $\delta(t)$,

$$\begin{aligned}\delta(t) &= (\rho\hat{p}/a) - u(x(t),x(t+1)) - \hat{p}(\rho x(t+1) - x(t)) \\ &= \rho\hat{p}(1 - y(t) - az(t+1)) + \mu\hat{p}(x(t) - y(t)),\end{aligned} \tag{2}$$

where $\mu\equiv(1-\rho(1-d))$. The aggregate value-loss of a program $\{x(t), y(t)\}$ is given by:

$$\sum_{t=0}^{\infty}\rho^t\delta(t) = \hat{p}[(\rho/a(1-\rho)) + x(0)] - \sum_{t=0}^{\infty}\rho^t u(x(t),x(t+1)). \tag{3}$$

It is clear from Eq. (3) that the optimality criterion reduces to one based on minimizing aggregate value-losses, since:

$$\sum_{t=0}^{\infty} \rho^t [u(x'(t), x'(t+1)) - u(x''(t), x''(t+1))] = \sum_{t=0}^{\infty} \rho^t [\delta'(t) - \delta''(t)] \tag{4}$$

where $\{x'(t), y'(t)\}$ and $\{x''(t), y''(t)\}$ are two programs from the same initial stock.

4. A continuous optimal policy function

It is standard to verify that there exists an optimal program $\{\bar{x}(t), \bar{y}(t)\}$ from any initial stock $x \in \mathbb{R}_+$. We define: $V(x) = \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$, and refer to V as the value function. It is easy to check that V is concave and non-decreasing on $\mathbb{R}_+$.

An elementary fact of the RSS model is that an optimal program $\{\bar{x}(t), \bar{y}(t)\}$ from x satisfies the full employment property: $\bar{y}(t) = 1 - a[\bar{x}(t+1) - (1-d)\bar{x}(t)]$ for all $t \in \mathbb{N}$ (see Khan and Mitra (2004b)). We use this to present a sufficient condition for the uniqueness of optimal programs. Our sufficient condition involves a restriction on the discount factor. The standard argument for uniqueness of optimal programs, relying on the strict concavity of the reduced-form utility function is not applicable in the RSS model.

Proposition 1. *For any economy E, there is a unique optimal program from every initial stock if $\rho < a$.*

Proof. If not, there exist two optimal programs $\{x(t), y(t)\}$ and $\{x'(t), y'(t)\}$ from some initial stock x, and for which, without any loss of generality, we may suppose that $x(1) > x'(1)$. If $y(1) = 0$, given the full employment property of optimal programs, it can be checked that the program $\{x(t), y(t)\}$ is dominated by a program that is identical to $\{x'(t), y'(t)\}$ except for the terms $(x, y(0))$ and $(x(1), (1 - a(x'(2) - (1-d)x(1))$. Thus, it remains only to consider the case where $y(1) > 0$. Given convexity of Ω and concavity of u, we may also suppose that the programs are such that $x(1)$ is "close enough" to $x'(1)$ so that $m \equiv (x(1) - x'(1)) < y(1)$.

Now consider a sequence that is identical to $\{x(t), y(t)\}$ except that its second term is given by $\{x'(1), y(1) - m\}$. It can be checked that this sequence is a program from x. Since the optimal program from $x'(1)$ will do at least as well as the program $(x'(1), x(2), x(3), \ldots)$, we obtain $V(x(1)) - V(x'(1)) \leq m$.

Since $\{x(t), y(t)\}$ and $\{x'(t), y'(t)\}$ are both optimal from x, we have $V(x) = y(0) + \rho V(x(1)) = y'(0) + \rho V(x'(1))$, which yields $\rho V(x(1)) - \rho V(x'(1)) = y'(0) - y(0) = am$, given the full employment property of optimal programs. Using the previous upper bound of ρm on this expression, we obtain $\rho \geq a$, and hence contradict our sufficient condition to complete the proof. □

Corollary 1. *If $\rho < a$, the optimal correspondence for the economy E is a continuous policy function h on the state space $X = [0, 1/ad]$.*

Proof. The optimal policy correspondence is actually a function in view of Proposition 1. The fact that it is continuous is a consequence of Berge's maximum theorem; see for example Dutta and Mitra (1989). □

Corollary 2. *If $\rho < a$ and $\{x'(t), y'(t)\}$ is an optimal program from 1, then for every program $\{x(t), y(t)\}$ with first two terms $\{(1, 1), (1-d, 1-d)\}$, we must have $x(2) \leq x'(2)$.*

Proof. Suppose there exists a program $\{x(t), y(t)\}$ with first two terms $\{(1, 1), (1-d, 1-d)\}$, and $x(2)>x'(2)$. Denote $u(x(0), x(1))+\rho u(x(1), x(2))$ by U and $u(x'(0), x'(1))+\rho u(x'(1), x'(2))$ by U'. Then $U=y(0)+\rho y(1)=1+\rho(1-d)$. Furthermore, $z'(1)=x'(1)+(1-d)\geq 0$, $y'(0)\leq 1-az'(1)$ and $y'(1)\leq x'(1)=z'(1)-(1-d)$. Thus, under the hypothesis $\rho<a$, we have $U'\leq 1+\rho(1-d)+z'(1)(\rho-a)\leq U$.

From the principle of optimality, we obtain $U+\rho^2 V(x(2))\leq U'+\rho^2 V(x'(2))$. With equality, we obtain two distinct optimal programs from $x=x(0)=x'(0)=1$, and contradict Proposition 1. With strict inequality, using the facts that $x(2)>x'(2)$, and V is non-decreasing, we obtain $U<U'$, a contradiction. □

5. Optimal topological chaos

In this section, we will be concerned with the dynamical system (X, h), where X and h are as specified in Corollary 1. The dynamical system (X, h) exhibits *topological chaos* if its *topological entropy* is positive [8]. It is known that if h (or an iterate of it) is *turbulent*, then (X, h) exhibits topological chaos. Since checking for turbulence is relatively easy, it is an especially useful *sufficient condition* for a dynamical system to exhibit topological chaos [9]. For our dynamical system, we show how this condition can be checked.

We assume a parametric regime in which a and d satisfy the restriction:

$$[1/(1+ad)]-[d(1-d)/a]=(1-d)^3 \tag{5}$$

Note that given any $d\in(0, 1)$, $[1/(1+ad)]\geq[1/(1+d)]>(1-d)$, which implies that as $a\rightarrow 1$, the left-hand side of Eq. (5) converges to a number greater than $(1-d)^2$, while it converges to $-\infty$ as $a\rightarrow 0$. Thus, by the intermediate value theorem, there is $a\in(0, 1)$ for which Eq. (5) holds, given any $d\in(0, 1)$.

We can now present the argument in two steps: first, we identify an optimal program under the parametric restriction Eq. (5); and second, we show that the dynamical system (X, h^2) is turbulent.

Proposition 2. *If $\rho<a$, and the parametric restriction Eq. (5) holds, the sequence $\{x(t), y(t)\}\equiv\{(1, 1), (1-d, 1-d), (\hat{x}/(1-d), 1), (\hat{x}, \hat{x}), (\hat{x}, \hat{x}), \cdots\}$ is an optimal program from Proposition 1.*

Proof. It can be checked that $\{x(t), y(t)\}$ is a program from Proposition 1; the only non-trivial step in this verification is to note that $x(2)=(1/a)-\xi(1-d)=\hat{x}/(1-d)$, by using Eq. (5). Suppose $\{x(t), y(t)\}$ is not optimal from Proposition 1. Denoting by $\{x'(t), y'(t)\}$ the optimal program from Proposition 1, and using Proposition 1, we have:

$$\sum_{t=0}^{\infty}\rho^t u(x(t),x(t+1))<\sum_{t=0}^{\infty}\rho^t u(x'(t),x'(t+1)) \tag{6}$$

Note that $\delta(t)=0$ for all $t\neq 2$. Substituting these values in Eqs. (2) and (3), and denoting $\hat{p}[(\rho/a(1-\rho))+x(0)]$ by α, we obtain: $\sum_{t=0}^{\infty}\rho^t u(x(t),x(t+1))=\alpha-\rho^2\delta(2)=\alpha-\rho^2\mu\hat{p}[x(2)-1]$. Since $y'(2)\leq 1$, a

[8] An alternative definition is that h has a periodic point of period that is not a power of 2. However, by a theorem due to Misiurewicz, these two definitions are equivalent. See Block and Coppel (1992, Proposition 34, p. 218) for an exposition of this result, as well as for a comprehensive discussion of the relevant concepts and definitions.

[9] This point has already been emphasized in Mitra (2001).

second appeal to Eqs. (2) and (3) yields: $\sum_{t=0}^{\infty}\rho^t u(x'(t),x'(t+1))\leq\alpha-\rho^2\delta'(2)\leq\alpha-\rho^2\mu\hat{p}[x'(2)-1]$. On making the necessary substitutions in Eq. (6), we obtain $x(2)>x'(2)$, and thereby contradict Corollary 2. This contradiction completes the proof. □

We can now present our principal result on topological chaos in the RSS model.[10]

Theorem 1. *Under the hypotheses of Proposition 2, the dynamical system (X, h) exhibits topological chaos.*

Proof. Given the optimal policy function h from Corollary 1, let $f(x)=h(h(x))\equiv h^2(x)$ for all x in X. Then, f is continuous on X. Consider the following three values of $x: A=\hat{x}=x(4)$, $B=\hat{x}/(1-d)=x(2)$ and $C=1=x(0)$, and note that (i) $f(B)=h(h(B))=h(\hat{x})=\hat{x}=A$; (ii) $f(A)=h(h(A))=h(\hat{x})=\hat{x}=A$; (iii) $f(C)=h(h(C))=h(x(1))=h(1-d)=\hat{x}/(1-d)=B$; (iv) $A=\hat{x}<1=C(\hat{x}/(1-d)=B$. In the form of a summary, we have:

$$f(B)=f(A)=A;\quad f(C)=B;\quad A<C<B$$

We can now assert that the function f is turbulent (Definition, p. 25), and therefore the topological entropy of f, $\psi\ (f)\geq\ \ln 2$ (Corollary 15, p. 200). This in turn implies that the topological entropy of h, $\psi(h)=(1/2)\psi(f)\geq\ \ln\sqrt{2}>0$, (Proposition 2, p. 191), so that (X, h) exhibits topological chaos. □

6. Concluding remarks

The point of this letter is not simply to provide yet another instance of topological chaos in an optimal growth model, but to do so in a simple setting where there is a widely-held presumption of monotonic convergence of optimal programs, having been established by Stiglitz in a continuous-time framework more than thirty six years ago. This being said, a precise delineation of the optimal policy correspondence of the RSS model for all values of the discount factor, and in particular to find values for which it is a unimodal function, remains an important open question stemming from the analysis presented here. Thus, the fact that the proof of Theorem 1 above does not appeal to the unimodal property, but only to that of continuity under the restriction of “small” discount factors, is surely of methodological import.

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[10] All page numbers in the proof to follow refer to the book by Block and Coppel (1992).

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